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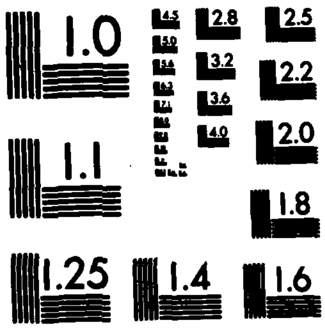

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detrapping events occur, even for an infinite wave train. A self-consistent theory for the emission process requires knowledge of the orbits throughout these different phases. One must follow the electrons for many trapping periods and accurately track the phase so that the detrapping point can be computed. The present report describes an asymptotic theory that allows this to be done, basically by finding an adiabatic invariant of the trapped electron motion. The calculation is done by an asymptotic ordering which relates the various small parameters in order to bring the appropriate physical processes into the expansion in a workable way. It then turns out that this ordering corresponds very well to the relations that occur in practice, with the numerical value of the basic expansion parameter being about .1 . This suggests that the analytic theory will have quite good accuracy and be useful for practical applications.

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MIT RESEARCH REPORT

ASYMPTOTIC ORBIT THEORY FOR TRAPPED ELECTRONS IN INHOMOGENEOUS WHISTLER FIELD

G. Hilfer and K. Molvig

ABSTRACT

The particle trapping phenomenon that occurs in the Whistler emission process is more complex than the trapping by an infinite wave in a homogeneous system. Because of the inhomogeneous background magnetic field and the antiparallel electron and wave velocities, electrons experience trapping for only a finite length of their orbits. Successive trapping and detrapping events occur, even for an infinite wave train. A self-consistent theory for the emission process requires knowledge of the orbits throughout these different phases. One must follow the electrons for many trapping periods and accurately track the phase so that the detrapping point can be computed. The present report describes an asymptotic theory that allows this to be done, basically by finding an adiabatic invariant of the trapped electron motion. The calculation is done by an asymptotic ordering which relates the various small parameters in order to bring the appropriate physical processes into the expansion in a workable way. It then turns out that this ordering corresponds very well to the relations that occur in practice, with the numerical value of the basic expansion parameter being about .1 . This suggests that the analytic theory will have quite good accuracy and be useful for practical applications.

I. INTRODUCTION

The report describes an asymptotic orbit calculation for high energy electrons moving along the earth's magnetic field in the presence of a large amplitude Whistler wave. This calculation is the necessary first step in the development of a self-consistent non-linear theory of the triggered emission process. An adiabatic invariant for the trapped particle motion is found from a multiple time scale analysis of the trajectories. This allows the orbits to be followed for many trapping periods to keep track of the phase and to compute the detrapping point. From this an expression for the nonlinear resonant current can be written down to, in principal, close the WKB wave equations describing the emission and propagation characteristics.

In section II, the orbit theory is developed as an asymptotic expansion to bring out the main physical characteristics. The resulting ordering is compared with typical experimental parameters in section III. Section IV reviews the properties of the orbital phase space, including an evaluation of the trapped particle separatrix.

II. ORBIT ANALYSIS

In the centered dipole approximation (see Helliwell's book Whistler and Related Jonospheric Phenomena, Appendix) the magnetic field strength is given by

$$B = B_o \left(\frac{R_o}{R} \right)^3 (1 + 3 \sin^2 \phi)^{1/2} \tag{1}$$

where ϕ is the geomagnetic latitude and R the geocentric radius which can be expressed as

$$R = R_o \frac{\cos^2 \phi}{\cos^2 \phi_o} \tag{2}$$

In the region of interest about the equator, i.e. for small ϕ, the field can be expressed as

$$B = B_o \left(1 + \frac{s^2}{L^2} \right) \tag{3}$$

s being the arclength along the line and L being the scale length of the gradient. In this field the electron equations of motion in the presence of the Whistler field are, [1]

$$\frac{ds}{dt} = v_\parallel \tag{4}$$

$$\frac{dv_\parallel}{dt} = -\Omega_o \frac{B^W}{B_o} V_\perp \cos\Psi - \frac{\mu}{m} \frac{\partial B}{\partial s} \tag{5}$$

$$\frac{d}{dt}\left(\frac{\mu}{m} \right) = \frac{V_\perp}{B} \left(V_\parallel - \frac{\omega}{k} \right) \Omega_o \frac{B^W}{B_o} \cos\Psi \tag{6}$$

$$\frac{d\Psi}{dt} = - \frac{\left(V_\parallel - \frac{\omega}{k}\right)}{V_\perp} \Omega_o \frac{B^W}{B_o} \sin \Psi + k v_\parallel - \omega + \Omega , \tag{7}$$

B^W stands for the wave magnetic field amplitude.

Where the electron gyrofrequency is given by

$$\Omega = \Omega_o\left(1 + \frac{s^2}{L^2}\right), \ \Omega_o = \frac{eB_o}{m_e c} \quad , \tag{8}$$

the magnetic moment μ is defined as $\mu = \dfrac{\frac{1}{2} mV_\perp^2}{B}$ (9)

where $v_\perp$ is the magnitude of the electronvelocity vector perpendicular to the earthmagnetic field direction. The phase ϕ of this vector has been replaced by the relative phase Ψ between $v_\perp$ and the electric field vector of the triggering wave:

$$\Psi = \int kds - \int \omega\, dt + \phi \quad . \tag{10}$$

In order to arrive at this set of differential equations for the behavior of an electron we have utilized the smallness of the gyroradius compared to the scale length of the magnetic field.

This makes μ in the absence of the wave a constant of the motion. In the presence of the wave both μ and the total particle energy will change. Adding equations (5) and (6) gives the exact expression for the change of total kinetic energy:

$$\frac{1}{m}\frac{dW}{dt} = \frac{d}{dt}\left(\frac{1}{2} v_\parallel^2 + \frac{1}{2} v_\perp^2\right) = -v_\perp \frac{\omega}{k} \ \Omega_o \frac{B^W}{B_o} \cos\Psi \tag{11}$$

Rigorously speaking everything on the right-hand side of equation (11), except for Ω_o and B_o , is a function of s and t . The evaluation of $v_\perp$ is described by equation (6) which can be written alternatively, as

$$\frac{dv_\perp}{dt} = \left(V_\parallel - \frac{\omega}{k}\right)\Omega_o \frac{B^W}{B_o} \cos\Psi + \frac{1}{2}\frac{V_\parallel V_\perp}{B}\frac{\partial B}{\partial s} \tag{12}$$

Furthermore, the phase velocity, $\frac{\omega}{k}$, of the wave will change as a function

of s and t and after the instability touches off B^w, the wave field, will grow in space and time.

While these changes occur on a slow time scale the angle Ψ will change very rapidly for most electrons. Since the propagation characteristics of the waves are linear (due to cold electrons) and the wavelength and period of oscillation are small compared to the spatial and temporal scales of evolution, the Whistler wave can be described by a WKB or eikonal form,

$$\vec{E} = \vec{P}\, E(s,t) e^{i\phi_w(s,t)} \tag{13}$$

where $\vec{P}$ is the polarization vector, E the wave amplitude and ϕ_w the wave phase. The last is determined from an integration of the ray equations

$$\frac{\partial \phi_w}{\partial s} = k \; ; \quad \frac{\partial \phi_w}{\partial t} = -\omega \tag{14}$$

Then the phase Ψ between the wave and the particles $\vec{V}_\perp$ becomes

$$\Psi = \phi_w(s,t) + \phi(s,t) \tag{15}$$

where the particles phase ϕ is evolving according to the equations of motion:

$$\frac{d\phi}{dt} = -\frac{\left(V_\parallel - \frac{\omega}{k}\right)}{V_\perp} \Omega_o \frac{B^w}{B_o} \sin\Psi + \Omega \tag{16}$$

the rate of change of Ψ is given by

$$\frac{d\Psi}{dt} = \frac{ds}{dt}\frac{\partial\phi_w}{\partial s} + \frac{\partial\phi_w}{\partial t} + \frac{d\phi}{dt}$$

$$= v_\parallel k - \omega - \frac{V_\parallel - \frac{\omega}{k}}{V_\perp} \Omega_o \frac{B^w}{B_o} \sin\Psi + \Omega \tag{17}$$

The homogeneous field calculation with its resonance velocity $V_{res} = \frac{\omega-\Omega}{k}$ and Helliwell's experiments show that relevant wave phase velocities $\frac{\omega}{k}$ are of the order of the parallel velocity of the group of electrons of interest. Of course the plasma under consideration is trapped in the mirror magnetic geometry of the earth's dipole field which lends itself to loss cone driven instabilities. However, observations indicate that the medium is only weakly unstable, linearly. This leads us to conclude that even though an inbalance between $v_{\parallel}$ and $v_{\perp}$ exists both can still be on the same order of magnitude. Putting all of this together says that the first term on the right of equation (7) will be of order ε^2 compared to the rest. And we can write

$$\frac{d\phi}{dt} = kv - \omega + \Omega + O(\varepsilon^2) \tag{21}$$

This says that for certain $v_{\parallel}$ the particles will not change their phase relation with respect to the wave field. This occurs for velocities near the resonant velocity

$$V_{res} = \frac{\omega-\Omega}{k} \tag{22}$$

It is convenient to define a new variable

$$v_{\parallel} = V_{res} + v \qquad \text{with } \frac{v}{V_{\parallel}} << 1 \tag{23}$$

and one can write

$$\frac{dv_{\parallel}}{dt} = \frac{dV_{res}}{dt} + \frac{dv}{dt} = V_{\parallel}\frac{\partial V_{res}}{\partial s} + \frac{\partial V_{res}}{\partial t} + \frac{dv}{dt} =$$

$$= V_{res}\frac{\partial V_{res}}{\partial s} + \frac{\partial V_{res}}{\partial t} + v\frac{\partial V_{res}}{\partial s} + \frac{dv}{dt} \tag{24}$$

The phenomenon of trapping is contained in equation (5) when the inhomogeneity term $-\frac{\mu}{m}\frac{\partial B}{\partial s} = -\frac{2\mu}{m} B_o \frac{s}{L^2}$ can become sufficiently small near the equator, $s = 0$, that the wave term $-\Omega_o \frac{B^w}{B_o} v_\perp \cos\Psi$ dominates and causes the electrons to oscillate about some stable phase at a frequency

$$\omega_{tr} = \left(\Omega_o \, k v_\perp \frac{B^w}{B_o}\right)^{1/2} \tag{18}$$

the so-called trapping frequency ω_{tr} for well trapped particles.

We now focus attention on the trapped particles, and develop an asymptotic ordering of parameters that permits an expansion of the electron orbits. We will show later that this ordering accurately describes the real parameters in the observed emission process. Re-writing ω_{tr} in terms of the equatorial gyroradius $\rho = \frac{v_\perp^{eq}}{\Omega_o}$

$$\omega_{tr} = \Omega_o \left(k\rho \frac{B^w}{B_o}\right)^{1/2} \tag{19}$$

one notes first that ω_{tr} is much smaller than Ω_o since $\frac{B^w}{B_o}$ is a very small number (in practice on the order of 10^{-5}). In practice $k\rho$ is order 1 and thus should remain unordered. Then we can define a smallness parameter ε as

$$\frac{\omega_{tr}}{\Omega_o} = \varepsilon \tag{20}$$

indicating at the same time that $\frac{B^w}{B_o} \sim \varepsilon^2$.

Since Ψ is intrinsically an order one quantity we find for the region of interest ($\frac{s}{L}$ sufficiently small so that both terms on the right-hand side of equation (5) can compete) $\frac{d\Psi}{dt} \sim \omega_{tr}\Psi$. Note that for untrapped, nonresonant particles the phase, Ψ, changes on the gyroperiod time scale and another expansion procedure would have to be used.

In terms of this new variable the equations of motion read

$$\frac{ds}{dt} = V_{res} + v \tag{25}$$

$$\frac{dv}{dt} = -V_\perp \frac{B^w}{B_o} \Omega_o \cos\Psi - \frac{1}{2}\frac{v_\perp^2}{B}\frac{\partial B}{\partial s} - V_{res}\frac{\partial V_{res}}{\partial s} - v\frac{\partial V_{res}}{\partial s} - \frac{\dot{\omega}}{k} \tag{26}$$

where from the definition of V_{res} $\quad \dfrac{\partial V_{res}}{\partial t} = \dfrac{\dot{\omega}}{k}$ (27)

was used.

$$\frac{dv_\perp}{dt} = \left(v - \frac{\Omega}{k}\right)\Omega_o \frac{B^w}{B_o}\cos\Psi + \frac{1}{2}\frac{V_\perp}{B}(V_{res} + v)\frac{\partial B}{\partial s} \tag{28}$$

$$\frac{d\Psi}{dt} = kv + O(\varepsilon^2) \tag{29}$$

For trapping to occur, we must have the inhomogeneity and wave terms in equation (26) comparable. The terms on the right-hand side of this equation scale successively like this:

$$v_\perp \; \Omega_o \; \frac{B_w}{B_o} \sim \frac{V_\perp^{\,2} s}{L^2} \sim v_\parallel \frac{\Omega_o}{k}\frac{s}{L^2} \sim \frac{v}{v_\parallel} v_\parallel \frac{\Omega_o}{k}\frac{s}{L} \sim \frac{1}{k}\frac{\delta\omega}{dt} \tag{30}$$

When treating a constant frequency wave train the last term $\frac{\dot{\omega}}{k}$ can be neglected since V_{res} is not an explicit function of time. The proportionality (30) can be re-expressed as:

$$\varepsilon^2 \sim \frac{B^w}{B_o} \sim \frac{\rho}{L}\frac{s}{L} \sim \frac{s}{L}\frac{1}{kL} \sim \frac{v}{v_\parallel}\frac{s}{L}\frac{1}{kL} \tag{31}$$

By assumption $\frac{v}{v_\parallel} \ll 1$ such that the last term in (31) is very small compared to its precessor $\frac{v_\parallel}{v_\perp} \sim 1$ was used again. Also using $k\rho \sim 1$ again it is true that $\frac{\rho}{L}\frac{s}{L} \sim \frac{s}{L}\frac{1}{kL}$. Then (31) demands essentially

$$\frac{s}{L}\frac{1}{kL} \sim \varepsilon^2 \tag{32}$$

Now, we also want the s motion to be on a slower time scale than the bouncing in the wave crests so that s is essentially forzen (adiabatically) while trapping motion ensues. Therefore

$$\frac{\dot{s}}{s} \sim \varepsilon^{\alpha}\,\omega_{tr} \quad \text{where } 0 < \alpha \leq 1 \tag{33}$$

has to be determined self consistently. $\frac{\dot{s}}{s}$ scales like $\frac{V_{\#}}{s} \sim \frac{\Omega}{k}\frac{1}{L}\frac{L}{s} \sim \varepsilon^{\alpha}\,\omega_{tr}$

$$\text{or} \quad \frac{1}{kL}\frac{L}{s} \sim \varepsilon^{1+\alpha} \tag{34}$$

From both (32) and (34) one can solve for

$$\frac{s}{L} \sim \varepsilon^{\left(\frac{1-\alpha}{2}\right)} \tag{35}$$

In order to be able to treat $B \sim B_o$ on both these scales as essentially constant we want *the first correction term* $\left(\frac{s}{L}\right)^2$ to come in on the next slower time scale $\left(\varepsilon^{\alpha}\right)^2$. Hence

$$\left(\frac{s}{L}\right)^2 \sim \varepsilon^{2\alpha} \tag{36}$$

Both (35) and (36) can be solved to give

$$\alpha = \frac{1}{3} \rightarrow \frac{s}{L} \sim \varepsilon^{1/3} \tag{37}$$

This completes the asymptotic ordering, using the small parameter $\varepsilon^{1/3} = (B^w/B_o)^{1/6}$.

The fast time scale for trapped particle is ω_{tr}^{-1} on which scale s is frozen. On this and the next slower time scale $\varepsilon^{1/3}\,\omega_{tr}^{-1}$, where s motion occurs, the magnetic field can be treated as constant. It follows from (32) that

$$\frac{\rho}{L} \sim \frac{1}{kL} \sim \varepsilon^{5/3} \tag{38}$$

and since v changes as Ψ on the fast time scale ω_{tr}^{-1}

$$\frac{\dot{v}}{\Omega_o V_\perp} \sim \frac{\omega_{tr}}{\Omega_o}\frac{v}{V_\perp} \sim \frac{\omega_{tr}}{\Omega}\frac{v}{V_\parallel} \sim \varepsilon^2 \quad \text{or}$$

$$\frac{v}{V_\parallel} \sim \varepsilon \tag{39}$$

Thus the corrections to $v_\parallel = V_{res}$ are coming in on third order in the expansion parameter $\varepsilon^{1/3}$, and are not needed for computing the motion along the magnetic lines. To lowest order the equations of motion for trapped particles become:

$$\frac{ds}{dt} = V_{res} + O(\varepsilon) \tag{40}$$

$$\frac{dv}{dt} = -v_\perp \frac{B^W}{B_o}\Omega_o \cos\Psi - \frac{1}{2}\frac{V_\perp^2}{B}\frac{\partial B}{\partial s} - V_{res}\frac{\partial V_{res}}{\partial S} - \frac{\dot{\omega}}{k} + O(\varepsilon) \tag{41}$$

$$\frac{dv_\perp}{dt} = 0 + O(\varepsilon^{2/3}) \tag{42}$$

$$\frac{d\Psi}{dt} = kv + O(\varepsilon^2) \tag{43}$$

In proceeding toward a solution of this set of equations it can be noted first that the time derivative of equation (43) becomes

$$\frac{d^2\Psi}{dt^2} = k\frac{dv}{dt} + v\frac{dk}{dt} \tag{44}$$

The last term in this equation scales like

$$v\dot{k} \sim v\, V_\parallel \frac{\partial k}{\partial s} \sim \frac{v}{V_\parallel}\,\Omega^2 \left(\frac{V_\parallel}{\Omega}\right)^2 \frac{ks}{L^2} \sim \frac{v}{V_\parallel}\,\Omega^2\, k_o \frac{s}{L}\frac{\rho}{L} \tag{45}$$

Comparing this to the left-hand side which scales like

$$\frac{d^2\Psi}{dt^2} - \omega_{tr}^2 \Psi \tag{46}$$

one finds

$$v\dot{k} - \varepsilon \frac{\Omega^2}{\omega_{tr}^2} \quad \omega_{tr}^2 \; 1\varepsilon^{1/3} \; \varepsilon^{5/3} - \varepsilon \, \omega_{tr}^2 \tag{47}$$

that this term is a third order correction in $\varepsilon^{1/3}$ leaving

$$\frac{d^2\Psi}{dt^2} = k \frac{dv}{dt} + O(\varepsilon) \tag{48}$$

Or with equation (41)

$$\frac{d^2\Psi}{dt^2} = -k \; v_\perp \frac{B^w}{B_o} \Omega_o \cos\Psi - \frac{1}{2}\frac{V_\perp^2}{B} k \frac{\partial B}{\partial s} - V_{res} \; k \frac{\partial V_{res}}{\partial s} - \dot{\omega} \tag{49}$$

One recognizes the first term on the right-hand side as the wave term, $\omega_{tr}^2 \cos\Psi$, and the others as inhomogeneity terms. We define an inhomogeneity factor, S , according to,

$$\omega_{tr}^2 \; S = \left(-\frac{1}{2}\frac{V_\perp^2}{B}\frac{\partial B}{\partial s} - V_{res}\frac{\partial V_{res}}{\partial s} - \frac{\dot{\omega}}{k}\right)k \tag{50}$$

Hence, *finally the* Ψ*-equation takes the form*

$$\frac{d^2\Psi}{dt^2} + \omega_{tr}^2 \cos\Psi = \omega_{tr}^2 \; S \tag{51}$$

There exist bounded oscillatory solutions to this equation as long as the inhomogeneity on the right-hand side satisfies

$$|S| \leq 1 \tag{52}$$

assuming for now that the frequency stays constant.

By the method of multiple time scales this equation can be solved analytically for trapped particles. Expanding in powers of $\varepsilon^{1/3}$:

$$\Psi = P_o + \Psi_1 + \Psi_2 + \ldots \tag{53}$$

$$S = S_o + S_3 + \ldots \tag{54}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t_o} + \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} + \ldots \tag{55}$$

Then

$$\frac{d^2\Psi}{dt^2} = \frac{\partial^2\Psi_o}{\partial t_o{}^2} + \varepsilon^{1/3}\left(\frac{\partial^2\Psi_1}{\partial t_o{}^2} + 2\frac{\partial^2\Psi_o}{\partial t_o\partial t_1}\right) + \varepsilon^{2/3}\left(\frac{\partial^2\Psi_2}{\partial t_o{}^2} + 2\frac{\partial^2\Psi_1}{\partial t_o\partial t_1}\right.$$

$$\left. + 2\frac{\partial^2\Psi_o}{\partial t_o\partial t_2} + \frac{\partial^2\Psi_o}{\partial t_1{}^2}\right) + \ldots \tag{56}$$

and $\cos\Psi = (\cos P_o) - (\Psi_1 \sin P_o) - \left(\frac{1}{2}\Psi_1^2 \cos P_o + \Psi_2 \sin P_o\right) + \ldots$ (57)

corrections to S will come in to third order. The stable phase P_o about which the trapped particles oscillate does not change on the ω_{tr}^{-1} time scale

$$\frac{\partial P_o}{\partial t_o} = 0 = \frac{\partial S_o}{\partial t_o} \tag{58}$$

neither does the inhomogeneity term. Thus leading order is

$$\omega_{tr}^2\ (\cos P_o - S) = 0 \qquad S < 1 \tag{59}$$

yielding $$P_o = P_o(s(t)) = \cos^{-1} S \ . \tag{60}$$

to next order one finds

$$\frac{\partial^2}{\partial t_o^2}\Psi_1 + \omega_{tr}^2\ \sin(-P_o)\ \Psi_1 = 0 \tag{61}$$

an oscillator equation with oscillation frequency

$$\omega_o = \omega_{tr}\sqrt{\sin(-P_o)} \quad . \tag{62}$$

Thus Ψ_1 becomes

$$\Psi_1 = A(t_1, t_2, \ldots) \sin\phi\,(t_o, t_1, t_2, \ldots) \ , \tag{63}$$

where
$$\frac{\partial\phi}{\partial t_o} = \omega_o \tag{64}$$

Since ϕ is inherently secular on the fast time scale one has to compute $\frac{d\phi}{dt}$ to requisite order prior to being able to evaluate ϕ . To find $\frac{\partial\phi}{\partial t_1}$ the next equation in the expansion hierarchy is needed. It reads

$$\frac{\partial^2\Psi_2}{\partial t_o} + \omega_o^2\Psi_2 = -2\frac{\partial^2\Psi_1}{\partial t_o \partial t_1} - 2\frac{\partial^2 P_o}{\partial t_o \partial t_2} - \frac{\partial^2 P_o}{\partial t_1{}^2} + \frac{1}{2}\Psi_1^2 \cos P_o$$
$$+ \Psi_2 \sin P_o \quad . \tag{65}$$

Our objective is to eliminate secularities of Ψ_2 on the fast time-scale $\omega_o{}^{-1}$. Based on equation (58) $\frac{\partial^2 P_o}{\partial t_o}$ vanishes eliminating the second term on the right-hand side of equation (65). Also the third and fifth term cannot drive the left-hand oscillator at frequency ω_o . The fourth term:

$$\frac{1}{2}(\cos P_o)\,A^2 \sin^2(\omega_o t + \ldots) = \frac{1}{4}A^2(\cos P_o)[1 - \cos(2\omega_o t + \ldots)] \tag{66}$$

is driving at twice the frequency ω_o, also dropping out of interest. It is then left to require that

$$\frac{\partial^2 \Psi_1}{\partial t_1 \partial t_o} = \frac{\partial}{\partial t_1} (A\omega_o \sin \phi) =$$

$$= \frac{\partial}{\partial t_1} (A\omega_o) \sin \phi + A\omega_o \frac{\partial \phi}{\partial t_1} \cos\phi \tag{67}$$

will not drive at ω_o leading to a fast time scale secularity in Ψ_2 . This will be guaranteed under the following two conditions:

$$\frac{\partial}{\partial t_1} A\omega_o = 0 \tag{68}$$

and

$$\frac{\partial \phi}{\partial t_1} = 0 \tag{69}$$

Therefore the combination $A\omega_o$, the amplitude of trapping oscillations times the frequency is constant on the slow time scale. This is a rather unexpected constant of the motion pertaining to the slow time evaluation of the trapped particle motion. And an additional fact is of great importance: equation (69), which allows us to integrate ϕ as follows

$$\phi = \int dt\ \omega_{tr} \sqrt{\sin[-p_o(s(t))]} \tag{70}$$

From the equations (40), (50), and (60) everything is known to compute ϕ . Assembling these results we found an analytic description of well trapped electrons.

$$\Psi = P_o + A \sin\phi \tag{71}$$

$$\phi = \int dt\ \omega_{tr} \sqrt{\sin(-p_o)} \tag{72}$$

With the adiabatic invariant

$$A\ \omega_o = \text{constant} \tag{73}$$

From equation (43) we find that the perturbed velocity v is of constant amplitude

$$v = \frac{A\omega_o}{k} \cos\phi$$

Summarizing these results the trapped particles are found to gyrate in vortices in v - Ψ - space on the fast timescale ω_{tr}^{-1}. Excursions in v are of constant amplitude. The average phase angle P_o is slowly changing as the particles are drifting further along the field lines. Outside the region of trapping $|S| \geq 1$, the particles follow adiabatic orbits in an inhomogeneous field to leading order. For simplicity an infinite wave train of constant amplitude was considered.

III. COMPARISON WITH EXPERIMENTAL PARAMETERS

After having developed the analytic expressions for trapped electron orbits the question arises, how do our assumptions compare with the measurements? Are the expansion parameters apt to validate the calculations?

In order to check the expansion against the data given in the literature our interest will focus on a field line designated as $\mathcal{L} = 4$.

The $\mathcal{L}$ number is the multiple of earth radii (R_o = 6371 km) that gives the straight distance from the center of the earth to the equator of the fieldline.

Granted a dipole representation of the earth magnetic field the $\mathcal{L} = 4$ line originates at $\phi_o = 60°$ latitude on the earth's surface. This corresponds to a fieldline length of

$$s = \frac{R_o}{\sqrt{3}\cos^2\phi_o}\left(\sinh^{-1}\left(\sqrt{3}\sin\phi_o\right)+\left(\sqrt{3}\sin\phi_o\right)\sqrt{1 + 3\sin^2\phi_o}\right) =$$

$$= 57{,}360 \text{ km} \tag{74}$$

(The arc length in this form was given by Chapman and Sugiura in 1956) [2]. At and in the vicinity of the equator of the fieldline under consideration, the magnetic field can be expressed as

$$B = B_o\left(1 + \frac{s^2}{L^2}\right) \quad . \tag{75}$$

The scale length is given by

$$L = \frac{\sqrt{2}\,\mathcal{L}\,R_o}{3} = 12{,}013 \text{ km} \tag{76}$$

For the proposed comparison of scales one needs to compute the relevant plasma parameters like Ω_o , ω_{pe} , V_{res} , ρ , etc. as well as wave parameters.

The dipole approximation to the earth magnetic field gives an electron gyrofrequency of $f_{ce} = 13.65$ kHz at an altitude of 3 earth radii above the earth's equator [2]. This number compares well with the more detailed analysis by Fougere using spherical harmonics which gives $f_{ce} = 14$ kHz [3]. The corresponding $\Omega_o = 2\pi f_{ce} = 88000$ rad/sec.

Typical observation are performed with the fieldstrength of the triggering signal lying in the range of 1 - 10 mγ [4]. The commonly used unit of γ is defined by $1\gamma = 10^{-9}$ Tesla $= 10^{-5}$ Gauss.

Hence:

$$\frac{B^W}{B_o} = \frac{5 \times 10^{-3}\gamma}{500\gamma} = 10^{-5} - \epsilon^2$$

This proves to be a very good expansion parameter. It is known experimentally that the frequency, where triggering of emission takes place most frequently, is half the minimum gyrofrequency [5] of electrons along the fieldline Ω_o :

$$\omega \simeq \frac{\Omega_o}{2} = 44000 \text{ rad/sec; } f = 7 \text{ kHz} \quad .$$

Helliwell's measurements of the "nose"-frequencies of risers and fallers allow to conclude on the plasma frequency [2] which he gives as

$$\omega_{pe} = 7.6 \times 10^5 \text{ rad/sec} \quad \text{corresponding to}$$

180 electron/cm^3 . Now we are in the position to compute the wavelength based on the linear dispersion relation for whistler mode signals in homogeneous magnetized plasmas:

$$\epsilon = 1 - \frac{c^2 k^2}{\omega^2} - \frac{\frac{\omega_{pe}^2}{\omega^2}}{1 - \frac{\omega_{ce}}{\omega}} = 0 \tag{77}$$

where ω_{ce} is the electron gyrofrequency. Plugging in the numbers k turns out to be

$$k = 2.5 \frac{1}{km} \text{ corresponding}$$

to a wave length λ of approximately $\lambda = \frac{2\pi}{k} = 2.5$ km . We also take the resonance velocity on the order of the particle velocities

$$V_{res} = \frac{\omega - \Omega}{k} = 17{,}340 \frac{km}{s} \sim v_{\parallel} \sim v_{\perp} \tag{78}$$

(This is also justified by a set of model parameters in a late paper of R.A. Helliwell and U.S. Inan [6]. From this the gyroradius ρ is 0.2 km, a tenth of the wave lenth. Then, very closely, $k\rho$ appears to be an order 1 quantity. The definition of the trapping frequency was

$$\omega_{tr}^2 = \Omega_o^2 \ K\rho \frac{B^w}{B_o} = 200 \frac{rad}{sec}$$

To within the accuracy that $k\rho$ is 1 it will be found that $\omega_{tr}/\Omega_o = \varepsilon$. Here ε is being defined by the square root of the ratio of the field strengths.

Next we are to compare the quantity $\frac{s}{L} \frac{1}{kL}$ with $\frac{B^w}{B_o}$ to find an upper bound. We can take the length L_p of the particle interaction region PIR as estimated by Helliwell [7]

$$L_p \simeq 800 \text{ km} \quad ,$$

or the arc length s up to which the effect of particle trapping can be found. I.E. s for which the two terms on the right-hand side of the equation of motion for $v_{\parallel}$ balance. The condition is

$$|S| = \left| \frac{s}{L} \left(\frac{2\Omega_o v_{\parallel}}{L\omega_{tr}^2} - \frac{k \ v_{\perp o}^2}{L \ \omega_{tr}^2} \right) \right| \leq 1 \tag{79}$$

from which follows that s can range up to 1500 km .

With Helliwell's estimate we find

$$\frac{s}{L}\frac{1}{kL} = 2.2 \times 10^{-6}$$

which is amasingly close to $\frac{B^W}{B_o} = 10^{-5}$, verifying our assumption

$$\frac{s}{L}\frac{1}{kL} \sim \frac{B}{B_o} \sim \varepsilon^2$$

Further we set $\frac{1}{s}\frac{ds}{dt} \sim \varepsilon^{\alpha}\,\omega_{tr}$. By definition $\frac{ds}{dt} = v_\parallel$. There follows

$$\frac{1}{s}\frac{ds}{dt} \simeq 21.7\,\frac{1}{s} = \frac{1}{9}\,\omega_{tr} \quad .$$

For our purposes of finding analytic expressions for the orbits, we expand hence in terms of $\varepsilon^{1/3} \sim \frac{1}{10}$. To conclude this section we give a short table of parameters relevant to the orbit calculations:

ω_{pe}^{eq}	= 760,000 rad/sec	L	= 12,013 km
Ω_o	= 88,000 rad/sec	λ	= 2.5 km
ω	= 44,000 rad/sec	k	= 2.5 1/km
ω_{tr}	= 200 rad/sec	ρ	= 0.2 km
$\frac{B^W}{B_o}$	= 10^{-5}	$V_\parallel \sim V_\perp \sim V_{res}$	= 17,440 km/s
$\frac{\omega_{tr}}{\Omega_o}$	= 2.2×10^{-3}	$\varepsilon^{1/3}$	~ 1/10
B_o	= 500γ	L	= 4
B^W	= 1 - 10 m γ	R_o	= 6,371 km
		s	~ 0 - 1500 km

IV. SEPARATRIX ANALYSIS

With the solutions to the equations of motion the wave-particle problem has been solved on a microscopic level. We found that the individual electron executes an orbit which is parametrized by two quantitites A and $\hat{\phi}$. A is the amplitude of trapping oscillation in Ψ and $\hat{\phi}$ the initial phase on this orbit, as it appears in the integration of ϕ (eq. 70).

Giving a pair of these parameters allows one to integrate the orbit back or forward to the entrapping or detrapping point. These points are determined by the intersection of the separatrix in $v - \Psi$ - space with the individual orbit. Exterior to those two points the electrons follow adiabatic orbits in the geomagnetic field to lowest order.

At exactly the equator, $S = 0$, the medium appears homogeneous and the phase space picture is as shown in Figure 1.

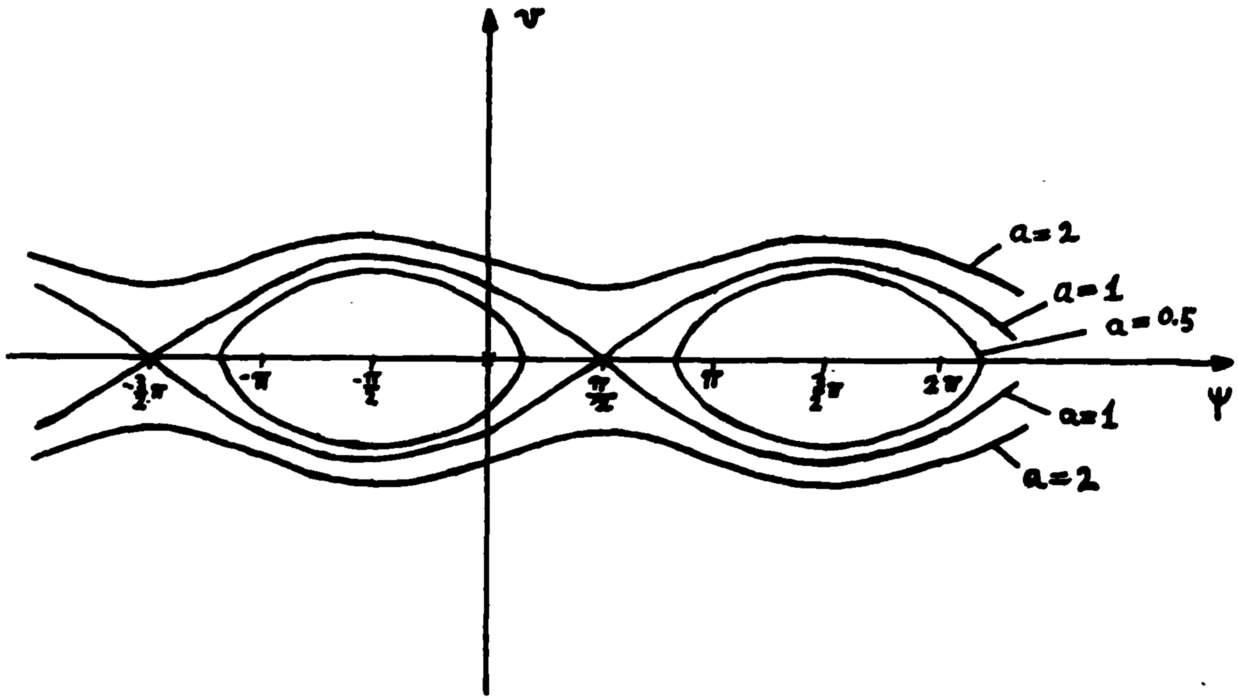

Figure 1.
Trajectories of cyclotron resonant electrons in the v-Ψ plane at the equator, S=0, or for the case of a homogeneous medium. The open-ended lines correspond to untrapped particles while the closed trajectories represent trapped particles. The separatrix is labeled by a=1 .

Clearly the closed line orbits of trapped electrons are separated from the open-ended lines for v greater than $\sqrt{2}\,\frac{\omega_{tr}}{k}$.

The equation for these orbit paths is

$$\frac{1}{2}\frac{k^2v^2}{\omega_{tr}^2} + \sin\Psi = a \tag{82}$$

Each path is labeled by a different constant of integration, a, of the first (energy) integral on the right-hand side of equation (82). For this constant being equal to 1 corresponds to the equation for the separatrix.

Away from the equator, the equation for the orbits is

$$\frac{1}{2}\left(\frac{kv}{\omega_{tr}}\right)^2 = a + S\Psi - \sin\Psi \tag{83}$$

Bounded orbits appear for $0 \le |S| \le 1$. Again the choice for the constant of integration, a, specifies each trajectory. A number of them are depicted in Figure 2. for $S = 0.5$.

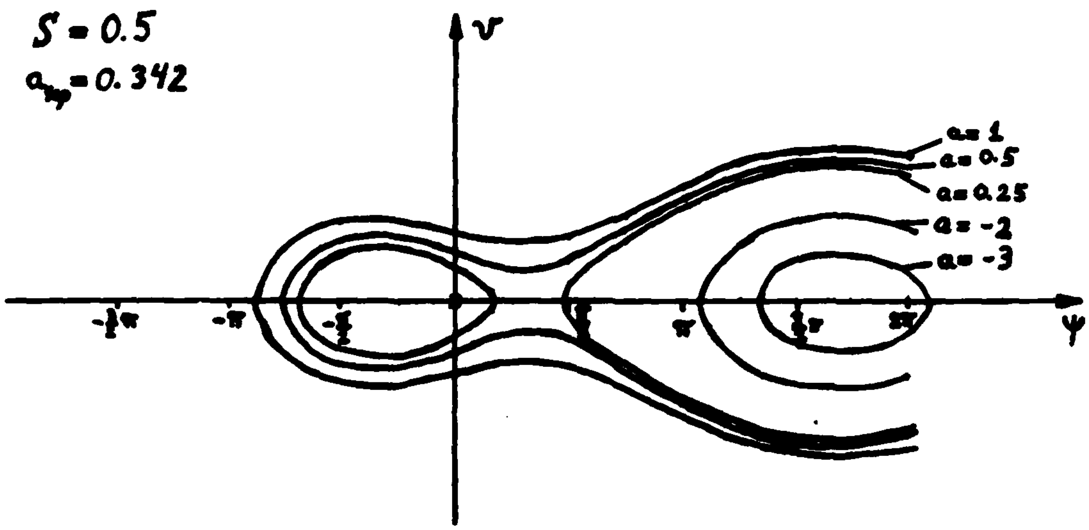

Figure 2.
Trajectories of cyclotron resonant electrons in the v-Ψ plane for S=0.5. The separatrix of the closed trajectories about P_o for $-\pi/2 < P_o < 0$ would be parametrized by a_{sep}=0.342.

The constant of integration, a, and its functional dependence on S can be found graphically and analytically after setting $v = 0$. This corresponds to the intersections of the contour lines with the Ψ-axis. The intersection values of Ψ, are the solutions to the transcendental equation

$$a + S\Psi = \sin\Psi \tag{84}$$

The straight line with slope S on the left equals the sine function on the right for several ψ's depending on the choice of a. This is illustrated by Figure 3.

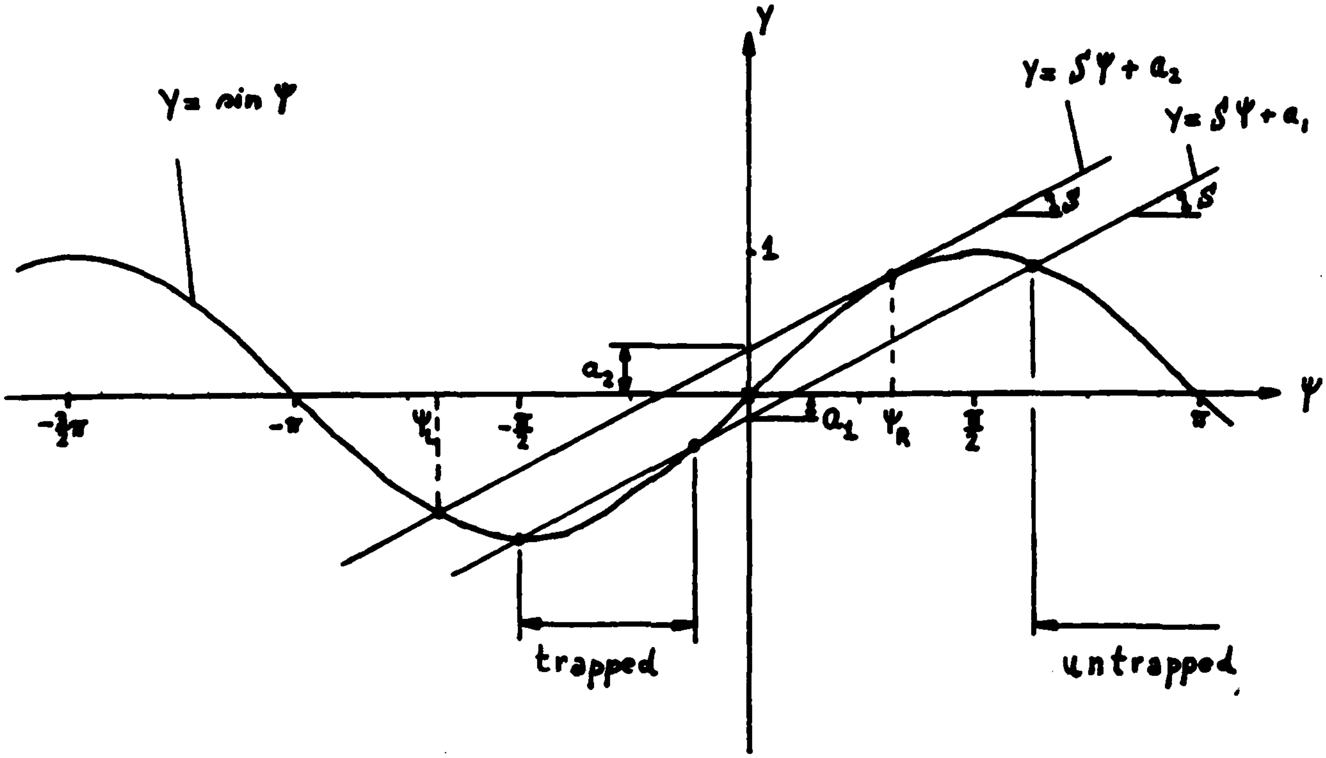

Figure 3.
Graphical determination of the constant of integration for the separatrix and its width in Ψ . See text for details.

Since Ψ is the angle between $v_\perp$ and the electric field vector of the wave it is sufficient to look at one interval of width 2π on the Ψ axis. For $0 < S \leq 1$ there will be maximally 3 intersections for small a's, two of which define the width in Ψ of the corresponding trapped particle orbit, the third one belongs to an open trajectory. The condition for the separatrix is that the open trajectory and the closed trajectory just touch. This happens for a_2 where the straight line $a_2 + S\Psi$ becomes tangential to sin Ψ (see Figure 3). The value Ψ_R where this happens is

$$\Psi_R = \cos^{-1} S \tag{85}$$

with $0 \le \Psi_R \le \pi/2$ for $S > 0$. The analog holds for $S < 0$. The value of a corresponding to the separatrix is given by

$$a_{sep} = S \cos^{-1} S - \sqrt{1 - S^2} \tag{86}$$

Hence, the equation for the separatrix about the trapped particle island centered about P_o , with $-\pi/2 \le P_o \le 0$, is

$$\frac{1}{2}\left(\frac{kv}{\omega_{tr}}\right)^2 = S \cos^{-1} S - \sqrt{1 - S^2} + S\Psi - \sin\Psi \tag{87}$$

This allows us to compute the entrapping point for nearly resonant electrons streaming into an infinite constant amplitude triggering wave. The equation to be evaluated is obtained by inserting

$$v = (A\omega_o) \cos\left(\int \omega_{tr} \sqrt{1 - S_{(t)}} \; dt\right) . \tag{88}$$

and

$$\Psi = \frac{(A\omega_o)}{\omega_{tr} \sqrt{1 - S^2(t)}} \sin\left(\int \omega_{tr} \sqrt{1 - S^2_{(t)}} \, dt\right) \tag{89}$$

into equation (85) and finding the root.

The separatrix has roughly the shape of an ellipse as can be seen from Figure 2. In order to find an analytic expression for its width in Ψ as a function of S it is necessary to reduce the transcendental character of the sine-function involved in equation (84).

It was found that a good approximation for sin Ψ_L is given by

$$-\frac{4}{\pi^2} \Psi_L^2 - \frac{12}{\pi} \Psi_L - 8 = a_{sep} - S\Psi_L \tag{90}$$

in the interval $-\frac{3\pi}{2} \le \Psi_L \le -\pi$ and

$$\frac{4}{\pi^2}\Psi^2_L + \frac{4}{\pi}\Psi_L = a_{sep} - S\Psi_L \qquad (91)$$

in the interval $-\pi \le \Psi_L \le 0$. The solution to either one of these equations gives the left-hand intersect of the separatrix contour with the Ψ axis. The right-hand intersect is given by equation (85). Therefore the width of the separatrix $\Delta\Psi_{sep} = \Psi_R - \Psi_L$ is given by

$$\Delta\Psi_{sep} = \cos^{-1} S + \begin{cases} \frac{3\pi}{2} + \frac{\pi^2}{8} S - \frac{\pi^2}{8}\sqrt{S + \frac{24}{\pi} S + \frac{16}{\pi^2}(1 - a_{sep})} \\ \\ \frac{\pi}{2} - \frac{\pi^2}{8} S + \frac{\pi^2}{8}\sqrt{S - \frac{8}{\pi} S + \frac{16}{\pi^2}\left(\frac{1}{2} + a_{sep}\right)} \end{cases} \qquad (92)$$

The upper expression is valid for the interval $-\frac{3\pi}{2} \le \Psi_L \le -\pi$, the lower expression for $-\pi \le \Psi_L \le 0$. The width in v of the separatrix Δv is given by

$$\Delta v = 2\,\frac{\omega_{tr}}{k}\sqrt{S\cos^{-1} S - \sqrt{1 - S^2}} \qquad (93)$$

So far S was treated as a positive quantity. It is straight forward to write down the expressions for $-1 \le S \le 0$.

Belabouring the separatrix to this extent is of importance. All of the interior are trapped particles which represent the dominant contribution to the resonant current density. This current in turn is responsible for the modifications of the triggering wave, the emission. The next step to be taken is to integrate over the resonant phase space region to compute this current. It will be to our advantage that the wave amplitude is growing on a slower time scale than the trapping motion.

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